Comparison of PML and ABC Formulations for Computational Acoustics in Unbounded Domains

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Introduction

One of the great challenges for wave propagation is the efficient and stable computation of wave in unbounded domains. Since the beginning of the eighties of the last century, several numerical techniques have been developed to deal with this topic: infinite elements, Dirichlet-to-Neumann operators based on truncated Fourier expansions, absorbing boundary conditions, etc. The advantages and drawbacks of these different approaches have been widely discussed in literature, see, e.g. [2]. We consider the second order wave equation in an unbounded domain and propose a recently developed perfectly matched layer (PML) technique for its efficient and reliable simulation. Furthermore, we consider the second order Engquist–Majda absorbing boundary condition, which we incorporated into the weak formulation of the acoustic wave equation in terms of Lagrange multipliers. In doing so, we concentrate on time domain formulations and use the Finite-Element (FE) method for the efficient numerical solution of the wave equation. Based on a numerical example, we will present a comparison of PML and ABC formulations.

PML Formulation

Applying a complex-valued coordinate stretching to the standard Helmholtz equation [4], results in

$$\eta_x\eta_y\eta_z \left(\frac{j\omega}{c}\right)^2 \hat{p} - \frac{\partial}{\partial x}\left(\frac{\eta_y\eta_z}{\eta_x}\frac{\partial \hat{p}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\eta_x\eta_z}{\eta_y}\frac{\partial \hat{p}}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\eta_x\eta_y}{\eta_z}\frac{\partial \hat{p}}{\partial z}\right) = 0 \qquad (1)$$

with

$$\eta_x = 1 + \frac{\sigma_x}{j\omega}\,; \quad \eta_y = 1 + \frac{\sigma_y}{j\omega}\,; \quad \eta_z = 1 + \frac{\sigma_z}{j\omega}\,. \qquad (2)$$

Here, $\hat{p}$ denotes the acoustic pressure, c the speed of sound, ω the pulsating frequency, and $\sigma_x, \sigma_z, \sigma_z$ the damping coefficients. The main point is now to introducing the scalar auxiliary variable v by $v = \hat{p}/(j\omega)$ and define the following relations between the vectorial auxiliary variable $\vec{u}$ and $\hat{p}$ as

$$\begin{aligned} u_x &= \frac{1}{\sigma_x + j\omega}\left(\sigma_y\sigma_z\frac{\partial v}{\partial x} + (\sigma_y + \sigma_z - \sigma_x)\frac{\partial \hat{p}}{\partial x}\right) \\ u_y &= \frac{1}{\sigma_y + j\omega}\left(\sigma_x\sigma_z\frac{\partial v}{\partial y} + (\sigma_x + \sigma_z - \sigma_y)\frac{\partial \hat{p}}{\partial y}\right) \\ u_z &= \frac{1}{\sigma_z + j\omega}\left(\sigma_x\sigma_y\frac{\partial v}{\partial z} + (\sigma_x + \sigma_y - \sigma_z)\frac{\partial \hat{p}}{\partial z}\right). \end{aligned} \qquad (3)$$

Therewith, the inverse Fourier transform of (1) using (3) achieve the following coupled system of partial differential equations

$$\frac{1}{c^2}\ddot{p} + \alpha\dot{p} + \beta p + \gamma v - \triangle p - \nabla\cdot\vec{u} = 0 \qquad (4)$$

$$\dot{\vec{u}} + A\,\vec{u} + B\,\nabla p - C\,\nabla v = 0 \qquad (5)$$

$$\dot{v} = p \qquad (6)$$

with

$$\alpha = \frac{\sigma_x + \sigma_y + \sigma_z}{c^2}\,; \;\; \beta = \frac{\sigma_x\,\sigma_y + \sigma_x\,\sigma_z + \sigma_y\,\sigma_z}{c^2}\,; \;\; \gamma = \frac{\sigma_x\sigma_y\sigma_z}{c^2}$$

$$A = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}; \;\; C = \begin{pmatrix} \sigma_y\sigma_z & 0 & 0 \\ 0 & \sigma_x\sigma_z & 0 \\ 0 & 0 & \sigma_x\sigma_y \end{pmatrix}$$

$$B = \begin{pmatrix} \sigma_x - \sigma_y - \sigma_z & 0 & 0 \\ 0 & \sigma_y - \sigma_x - \sigma_z & 0 \\ 0 & 0 & \sigma_z - \sigma_x - \sigma_y \end{pmatrix}.$$

ABC Formulation

The first order Engquist–Majda absorbing boundary condition, which allows to substitute the normal derivate of the acoustic pressure by $(1/c)\,\partial p/\partial t$ [1] can be easily in-cooperated into the weak formulation of the wave equation and results in almost no addition computational costs (see, e.g., [3]). However, the second order Engquist–Majda absorbing boundary condition reads as follows [1]

$$c^{-1}\frac{\partial \dot{p}}{\partial \vec{n}} + c^{-2}\ddot{p} - \frac{1}{2}p_{\tau\tau} = 0 \qquad (7)$$

with $p_{\tau\tau}$ the 2nd derivative of the acoustic pressure with respect to the tangential direction. We incorporated this relation into the weak formulation of the linear wave equation

$$\int_\Omega c^{-2}\ddot{p}\phi\; d\Omega + \int_\Omega (\nabla p\cdot\nabla\phi)\; d\Omega - \int_{\Gamma_\mathrm{A}} \phi\nabla p\cdot\vec{n}\; d\Gamma \qquad (8)$$

with ϕ being an appropriate test function and Γ_A the outer boundary of the computational domain Ω. First of all, we introduce a Lagrange multiplier $\Lambda = -\nabla p\cdot\vec{n}$ on the boundary Γ_A. In terms of Λ, we can rewrite the boundary integral as

$$-\int_{\Gamma_A} \phi\nabla p\cdot\vec{n}\; d\Gamma = \int_{\Gamma_\mathrm{A}} \phi\Lambda\; d\Gamma. \qquad (9)$$

Moreover, such a substitution gives the possibility to reformulate the ABCs (7) in the form

$$-c^{-1}\dot{\Lambda} + c^{-2}\ddot{p} - \frac{1}{2}p_{\tau\tau} = 0\,. \tag{10}$$

A weak formulation of (10) and following integration by parts yield (in 2D)

$$\int_{\Gamma_A} \left(\left(-c^{-1}\dot{\Lambda} + c^{-2}\ddot{p} \right) \mu + \frac{1}{2}p_\tau \mu_\tau \right) d\Gamma - \frac{1}{2}\mu p_\tau \Big|_{\partial\Gamma_A} = 0\,,$$

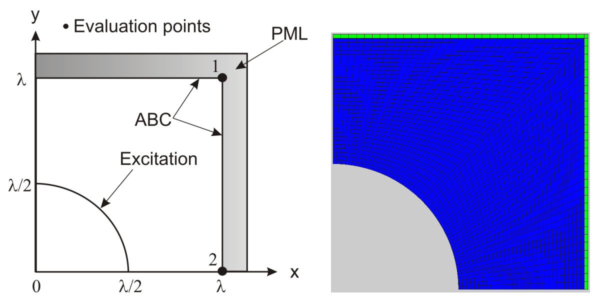
for all test functions μ out of an appropriate Lagrange multiplier space. Besides, here $\partial\Gamma_A$ represents the endpoints of the boundary Γ_A. Thus we have a system of two equations to be solved

$$\int_\Omega c^{-2}\ddot{p}\phi \, d\Omega + \int_\Omega (\nabla p \cdot \nabla \phi)\, d\Omega + \int_{\Gamma_A} \phi \Lambda \, d\Gamma = 0\,, \tag{11}$$

$$\int_{\Gamma_A} \left(\left(-c^{-1}\dot{\Lambda} + c^{-2}\ddot{p} \right) \mu + \frac{1}{2}p_\tau \mu_\tau \right) d\Gamma - \frac{1}{2}\mu p_\tau \Big|_{\partial\Gamma_A} = 0\,.$$

Numerical Tests

For the two formulations with PML (see (4)-(6)) and with ABC (see (11)) we apply standard Lagrangian finite elements and use the implicit Newmark scheme for time discretization. To compare both formulations we perform numerical computations of the setup displayed in Fig. 1. On the inner boundary we apply a cw-excitation with



(a) (b)

Figure 1: Geometric setup and FE grid: (a) Geometric setup; (b) FE grid (with one additional FE layer for PML).

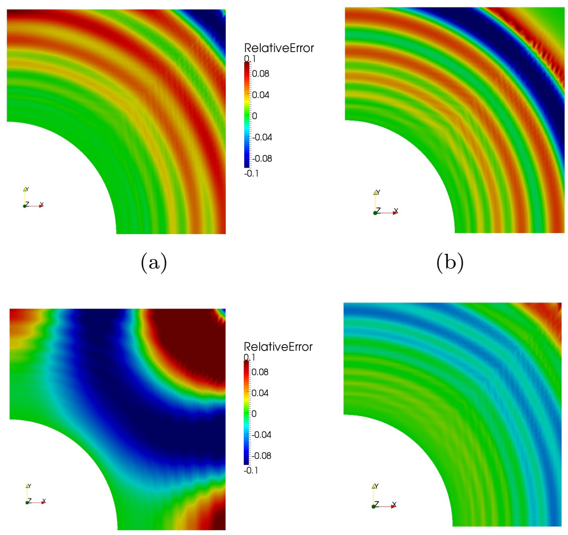
prescribed amplitude and frequency. The space discretization has been chosen to be quite fine with $h \approx \lambda/60$ (λ denotes the wavelength). For the time discretization we set $\Delta t = T/20$ with T the time period of the cw-signal. At the boundary of the computational domain we apply in a first step 1st and 2nd order ABCs and record acoustic pressure at the two evaluation points (see Fig. 1). In a second step, we extend the propagation domain by one, two till five additional finite element layers, in which we apply our PML formulation.

Since the considered configuration has an analytical solution (see, e.g., [3]), we can directly compute the relative error in the amplitude at the two evaluation points, which

Tabelle 1: Relative Error in the two evaluation points

Method	Eval. point 1	Eval. point 2
ABC 1st order	26.5%	6.1 %
ABC 2nd order	11.3%	7.5 %
PML 1 element	10.3%	0.38 %
PML 2 element	9.4%	0.82 %
PML 3 element	6.4%	0.48 %
PML 4 element	4.8%	0.17 %
PML 5 element	3.9%	0.05 %

is listed in Tab. 1. Furthermore, we visualize in Fig. 2 the computed relative amplitude error for two characteristic time steps: time step 18, where the wave impinges for the first time the boundary, and time step 50, where we already have multiple reflections. One can clearly notice, that the imperfection of 2nd order ABC results in a quite strong error in the propagation domain as compared to the PML technique (one finite element in thickness direction).



(c) (d)

Figure 2: Error plot: (a) ABC at time step 28; (b) PML at time step 28; (c) ABC at time step 50; (d) PML at time step 50.

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