The Lepskiĭ Principle for an Inverse Source Problem in Experimental Aeroacoustics

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Introduction

This conference contribution is concerned with variational inverse methods (i.e. generalized Tikhonov regularization) in experimental aeroacoustics. Such techniques aim at a reconstruction of the source power function of an aeroacoustic source from measured correlation data of an array of microphones.

More precisely we will investigate a Tikhonov functional composed of a quadratic data misfit functional and a sparsity promoting penalty functional. For an optimal accuracy of the reconstruction, the choice of the regularization parameter is crucial. Depending on the form of the penalty functional, there is a large number of strategies for the parameter selection. Possible approaches that have also been used for the aeroacoustic inverse problem are for instance the L-curve criterion [5, Section 7.5] or the Bayesian criterion [6]. This contribution deals with another parameter choice strategy that is well understood from a mathematical point of view - the Lepskiĭ principle (see e.g. [1, 4]). We will present a procedure, how the Lepskiĭ principle can be applied to the aeroacoustic inverse source problem. This procedure can also be applied to other penalty functionals than the one presented in this paper.

We would like to emphasize that the aim of this work is the demonstration of a framework that allows the application of the Lepskiĭ principle for the aeroacoustic inverse problem. In particular, a comparison with other parameter selection strategies is out of scope.

Sound propagation model

The standard sound propagation model in experimental aeroacoustics is the convected Helmholtz equation

$$(k + i\mathbf{m}\cdot\nabla)^2 p + \Delta p = -s \ . \tag{1}$$

Here k denotes the wavenumber, $\mathbf{m} \in \mathbb{R}^d$ with $|\mathbf{m}| < 1$ the constant Mach vector, p the sound pressure field in frequency domain and s a generic source. The free space Green's function of Equation (1) for $d = 3$ is given by

$$g(\mathbf{x},\mathbf{y}) = \frac{\exp\left(\frac{ik}{1-|\mathbf{m}|^2}\left(-(\mathbf{x}-\mathbf{y})\cdot\mathbf{m} + |\mathbf{x}-\mathbf{y}|_{\mathbf{m}}\right)\right)}{4\pi|\mathbf{x}-\mathbf{y}|_{\mathbf{m}}} \ ,$$

with the Mach norm

$$|\mathbf{z}|_{\mathbf{m}} = \sqrt{(\mathbf{z}\cdot\mathbf{m})^2 + \beta^2|\mathbf{z}|^2} \ , \tag{2}$$

and $\beta^2 = 1 - |\mathbf{m}|^2$.

Discrete forward model

Note that we consider array imaging methods here i.e. the sound pressure signals are recorded by an array of microphones. The positions of the M microphones in the array are denoted by $\mathbf{x}_1, \dots \mathbf{x}_M$. The source region (i.e. the region where one assumes the source power to be greater than zero) is furthermore discretized by N focus positions $\mathbf{y}_1, \dots, \mathbf{y}_N$. For a fixed wavenumber k we then get the pressure signal at the array

$$\mathbf{p} = \begin{pmatrix} p(\mathbf{x}_1) \\ \vdots \\ p(\mathbf{x}_M) \end{pmatrix} \tag{3}$$

and the source signal in the source region

$$\mathbf{s} = \begin{pmatrix} s(\mathbf{y}_1) \\ \vdots \\ s(\mathbf{y}_N) \end{pmatrix} \ . \tag{4}$$

Since the measurement environment is usually subject to random processes, both quantities are considered as random variables. With the propagation matrix, defined via the Green's function

$$\mathbf{G}_{ij} = g(\mathbf{x}_i, \mathbf{y}_j) \ , \tag{5}$$

we assume the following linear relationship between source and array data

$$\mathbf{p} = \mathbf{G}\mathbf{s} \ .$$

Taking the correlation matrix (a.k.a. cross spectral matrix CSM) of the array vector $\mathbf{p}$ yields the following forward model

$$\mathbf{C} := \mathbb{E}\{\mathbf{p}\mathbf{p}^*\} = \mathbf{G}\ \mathbb{E}\{\mathbf{s}\mathbf{s}^*\}\ \mathbf{G}^* \ . \tag{6}$$

A standard assumption in experimental aeroacoustics is that the source signal vector $\mathbf{s}$ is uncorrelated. Using this assumption Equation (6) can be reformulated by

$$\mathbf{C} = \mathbf{G}\ \mathbf{M}_{\mathbf{q}}\ \mathbf{G}^*, \tag{7}$$

with

$$\mathbf{q} = \left(\mathbb{E}\left\{|\mathbf{s}_1|^2\right\}, \dots, \mathbb{E}\left\{|\mathbf{s}_N|^2\right\}\right)^\top \quad \mathbf{M}_{\mathbf{q}} = \operatorname{diag}(\mathbf{q}_1, \dots, \mathbf{q}_N) \ .$$

The inverse problem is then to recover the source power vector $\mathbf{q}$ from given correlation data $\mathbf{C}$.

From now on we consider a vectorized formulation using the column-wise vectorization

$$\operatorname{vect}(\mathbf{A}) = (\mathbf{A}_{11}, \dots, \mathbf{A}_{M1}, \mathbf{A}_{12}, \dots, \mathbf{A}_{M2}, \dots, \mathbf{A}_{1d}, \dots, \mathbf{A}_{MM})^\top \ .$$

Hence we get the equivalent formulation of Equation (7)

$$\mathbf{C} = \mathbf{G}\ \mathbf{M}_{\mathbf{q}}\ \mathbf{G}^* \Leftrightarrow \operatorname{vect}(\mathbf{C}) = \mathbf{T}\mathbf{q} \ .$$

Where the $n-$th column of $\mathbf{T}$ is given by

$$[\mathbf{T}]_n = \overline{[\mathbf{G}]_n} \otimes [\mathbf{G}]_n$$

and $'\otimes'$ denotes the Kronecker product.

Experimental data
In a real experimental setup we usually do not have access to the exact CSM but only to a random approximation

$$\mathbf{C}^{\text{obs}} \approx \mathbf{C} = \mathbb{E}\{\mathbf{p}\mathbf{p}^*\} \ .$$

For the further procedure the covariance matrix of correlation data

$$\mathbf{\Sigma} = \text{Cov}\left(\text{vect}\left(\mathbf{C}^{\text{obs}}\right)\right) \in \mathbb{C}^{M^2 \times M^2} \ ,$$

will be of great importance. Under further statistical assumptions (zero mean Gaussian pressure data cf. [3]) the entries of $\mathbf{\Sigma}$ can be efficiently estimated by Isserlis' formula

$$\begin{aligned} &\text{Cov}\left(p(\mathbf{x}_m)\overline{p(\mathbf{x}_l)}, p(\mathbf{x}_{m'})\overline{p(\mathbf{x}_{l'})}\right) \\ &= \mathbb{E}\left(p(\mathbf{x}_m)\overline{p(\mathbf{x}_{m'})}\right)\overline{\mathbb{E}\left(p(\mathbf{x}_l)\overline{p(\mathbf{x}_{l'})}\right)} \\ &+ \mathbb{E}\left(p(\mathbf{x}_m)p(\mathbf{x}_{l'})\right)\overline{\mathbb{E}\left(p(\mathbf{x}_l)p(\mathbf{x}_{m'})\right)} \ . \end{aligned} \tag{8}$$

From now on we assume that the covariance matrix $\mathbf{\Sigma}$ or at least an estimation is available.

Generalized Tikhonov regularization

The generic formulation of generalized Tikhonov regularization is given by the following minimization problem

$$\frac{1}{2}\|\mathbf{T}\mathbf{q} - \text{vect}\left(\mathbf{C}^{\text{obs}}\right)\|_2^2 + \alpha\,\mathcal{R}(\mathbf{q}) \to \min \ .$$

Here $\mathcal{R}$ denotes a convex penalty functional and $\alpha > 0$ the regularization parameter. Typical choices for $\mathcal{R}$ may be

- $p-$norm powers for $p \geq 1$

$$\mathcal{R}(\mathbf{q}) = \frac{1}{p}\|\mathbf{q}\|_p^p \ ,$$

- box constraints with bounds $a < b$

$$\mathcal{R}(\mathbf{q}) = \begin{cases} 0 & \text{if } a \leq \mathbf{q}_n \leq b \text{ for all } n \\ \infty & \text{else} \end{cases} \ .$$

For our investigations we consider the following specific penalty functional

$$\mathcal{R}(\mathbf{q}) = \chi_{\mathcal{N}_+}(\mathbf{q}) + \frac{1-\theta}{2}\|\mathbf{q}\|_2^2 + \theta\|\mathbf{q}\|_1 \ . \tag{9}$$

With the nonnegativity constraint

$$\mathcal{X}_{\mathcal{N}_+}(\mathbf{q}) = \begin{cases} 0 & \text{if } \mathbf{q}_n \geq 0 \text{ for all } n \\ \infty & \text{else} \end{cases}$$

and $\theta \in [0,1]$ chosen a-priori. With this setup we can define the reconstruction from exact data

$$\mathbf{q}_\alpha := \underset{\mathbf{q}}{\text{argmin}} \frac{1}{2}\|\mathbf{T}\mathbf{q} - \text{vect}\left(\mathbf{C}\right)\|_2^2 + \alpha\,\mathcal{R}(\mathbf{q}) \tag{10}$$

and the reconstruction from noisy data

$$\hat{\mathbf{q}}_\alpha := \underset{\mathbf{q}}{\text{argmin}} \frac{1}{2}\|\mathbf{T}\mathbf{q} - \text{vect}\left(\mathbf{C}^{\text{obs}}\right)\|_2^2 + \alpha\,\mathcal{R}(\mathbf{q}) \ . \tag{11}$$

Application of the Lepskiĭ principle

Let denote $\mathbf{q}$ the ground truth source power vector. The idea of the Lepskiĭ principle relies on the following error decomposition

$$\|\mathbf{q} - \hat{\mathbf{q}}_\alpha\|_2 \leq \|\mathbf{q} - \mathbf{q}_\alpha\|_2 + \|\mathbf{q}_\alpha - \hat{\mathbf{q}}_\alpha\|_2 \ .$$

The first term of the upper bound is deterministic and does not depend on the noise - this is usually called the approximation error. The second term depends on the noise and is usually called the propagated data noise error. The strategy is now to roughly balance those two error contributions. Therefore, we consider a finite sequence of increasing values for the regularization parameter

$$\alpha_1 < \cdots < \alpha_L \ .$$

The important tool for the implementation of the Lepskiĭ principle is a decreasing function on the corresponding index set $\Psi : \{1, \ldots, L\} \to [0, \infty)$ that satisfies

$$\mathbb{E}\left(\|\mathbf{q}_{\alpha_j} - \hat{\mathbf{q}}_{\alpha_j}\|_2^2\right) \leq \Psi^2(j)$$

for all $j = 1, \ldots, L$. For our specific penalty functional (cf. Equation (9)) we cannot compute the upper bound Ψ explicitly but we can at least estimate it with a Monte-Carlo simulation based on the noise covariance $\mathbf{\Sigma}$. This procedure will be described in the next paragraph. Assuming for a moment that an appropriate bound Ψ is given, then the Lepskiĭ principle choses the regularization parameter according to the following index choice rule (cf. [4])

$$j^{\text{Lep}} := \max\left\{j : \ \|\hat{\mathbf{q}}_{\alpha_i} - \hat{\mathbf{q}}_{\alpha_j}\|_2 \leq 4\Psi(i) \text{ for all } i \leq j\right\} \ .$$

Note that this selection criterion does only depend on noisy reconstructions and the upper bound Ψ.

Estimation of the propagated data noise error
We will now describe in more detail how the function Ψ can be estimated. The proposed Monte-Carlo procedure relies on a principle of source power vector sampling that is taken from [2]. The sampling method generates samples of source power vectors $\mathbf{q}$ with n_{source} nonzero entries i.e.

$$\mathbf{q}_{k_l} = \mu_l \qquad \text{for } l = 1, \ldots, n_{\text{source}} \ .$$

The parameters $n_{\text{source}}, k_l, \mu_l$ are then sampled in the following manner

- $n_{\text{source}} \sim 1 + \text{Pois}(\lambda)$ where Pois denotes the Poisson distribution and $[1+\lambda]$ should be chosen in the range of the expected number of sources.
- k_l is sampled from a uniform distribution on a finer grid than the reconstruction grid in order to prevent inverse crimes. The forward matrix corresponding to the finer grid will be denoted by $\mathbf{T}^{\text{fine}}$.
- μ_l is sampled from a Rayleigh distribution $\text{Ray}(\sigma)$, with σ in the range of the expected average source power level.

Now we can sample n_{samp} source power vectors by the procedure above and for each $n = 1, \dots n_{\text{samp}}$ perform the following computations.

1. sample source power vector $\mathbf{q}^{(n)}$
2. set exact data: $\text{vect}\left(\mathbf{C}^{(n)}\right) := \mathbf{T}^{\text{fine}}\mathbf{q}^{(n)}$
3. sample noise $\mathbf{z}^{(n)}$ from $\mathcal{N}_{\mathbb{C}}\left(0, \mathbf{\Sigma}\right)$
4. set noisy data: $\text{vect}\left(\mathbf{C}^{\text{obs}(n)}\right) := \text{vect}\left(\mathbf{C}^{(n)}\right) + \mathbf{z}^{(n)}$
5. compute reconstructions

$$\mathbf{q}_{\alpha_j}^{(n)} := \underset{\mathbf{q}}{\text{argmin}} \frac{1}{2}\|\mathbf{T}\mathbf{q} - \text{vect}(\mathbf{C}^{(n)})\|_2^2 + \alpha_j \mathcal{R}(\mathbf{q}) \ ,$$

$$\hat{\mathbf{q}}_{\alpha_j}^{(n)} := \underset{\mathbf{q}}{\text{argmin}} \frac{1}{2}\|\mathbf{T}\mathbf{q} - \text{vect}(\mathbf{C}^{\text{obs}(n)})\|_2^2 + \alpha_j \mathcal{R}(\mathbf{q})$$

Note that for the generation of noisy data, we add a vector $\mathbf{z}$ sampled from a complex normal distribution with covariance structure $\mathbf{\Sigma}$.
The upper bound on the propagated data noise error is then estimated by

$$\Psi^{\text{est}}(j)^2 := \frac{1}{n_{\text{samp}}} \sum_{n=1}^{n_{\text{samp}}} \|\mathbf{q}_{\alpha_j}^{(n)} - \hat{\mathbf{q}}_{\alpha_j}^{(n)}\|_2^2 \qquad (12)$$

for $j = 1, \dots, L$.

Numerical example

To illustrate the procedure described previously we apply generalized Tikhonov regularization together with the Lepskiĭ principle to an experimental dataset. We consider a measurement of a Dornier 728 half-model [7] which is widely used as an experimental benchmark dataset (see e.g. [8, 9, 10]). The data can be found at [11] under the label 'DLR1'.

Computational settings

For the computations, only the lower triangular part of the CSM without the diagonal is used (i.e. only entries $\mathbf{C}_{ml}^{\text{obs}}$ with $m > l$). This applies to the reconstructions during the Monte Carlo routine as well as to the reconstruction on the experimental data. Therefore, the forward matrix $\mathbf{T}$, the vectorization operator $\text{vect}(\cdot)$ and the data covariance $\mathbf{\Sigma}$ are adapted accordingly. The remaining vectorized correlation data will be denoted by $\mathbf{y}^{\text{cor}}$. The sequence of regularization regularization parameters were scaled according to the correlation data i.e.

$$\alpha = \alpha_{\text{rel}} \cdot \|\mathbf{y}^{\text{cor}}\|_2^2 \ , \qquad (13)$$

where $\alpha_{\text{rel}} \in \{10^{-7+\frac{j-1}{3}} : \ j = 1, \cdots, 25\}$.

For the Monte Carlo routine that provides Ψ^{est} (cf. Equation (12)), the sources are sampled uniformly on a planar grid with resolution $\Delta_{xy} = 0.001\,\text{m}$. Furthermore, $n_{\text{source}} \sim 1 + \text{Pois}(4)$ and $\mu_s \sim \text{Ray}(0.2)$. The number of source vector samples is set to $n_{\text{samp}} = 100$.

The solution of the minimization problems arising from generalized Tikhonov regularization are computed using the generalized version of the FISTA algorithm (see [12, p.291]) with 1000 iterations.

Reconstruction results

We consider source power reconstructions for $\theta \in \{0, 0.5, 1\}$ on a planar grid with resoulution $\Delta_{xy} = 0.02\,\text{m}$. The microphone array consists of $M = 134$ microphones.

Figures 1,2 and 3 show power spectral density (PSD) source maps for the frequency 6000 Hz. Firstly we observe, that the penalty functionals with active L^1 penalty (i.e. $\theta = 0.5$ and $\theta = 1$) act strongly sparsity promoting. This is the well-known sparsity promoting effect of the L^1 regularization. As this effect acts quite strongly one should choose θ not to close to 1 to avoid the suppression of minor sources. However, the purely L^2 regularization ($\theta = 0$) shows strong noise effects. Thus, at first glance, a penalty functional with some sparsity promoting components ($\theta > 0$ but not too large) seems reasonable here.

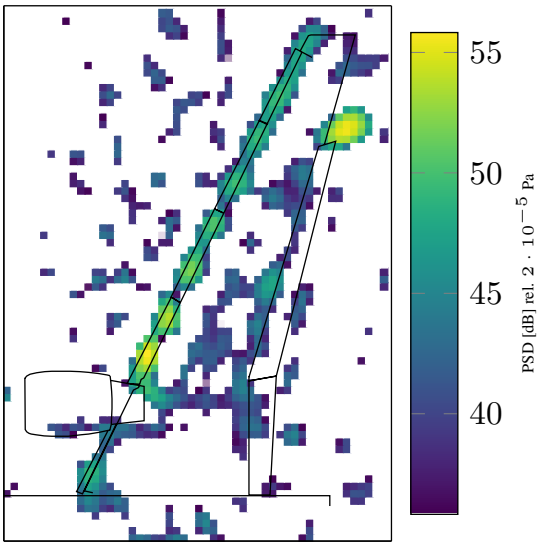

Figure 1: $\theta = 0,\ j^{\text{Lep}} = 22, \alpha_{\text{rel}} = 1$

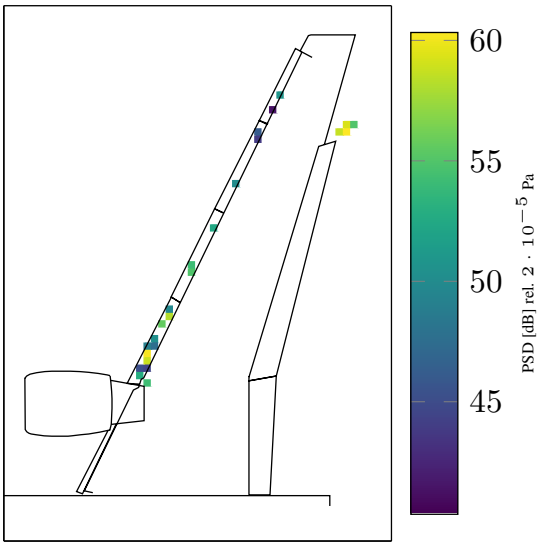

Figure 2: $\theta = 0.5,\ j^{\text{Lep}} = 20, \alpha_{\text{rel}} = 0.215$

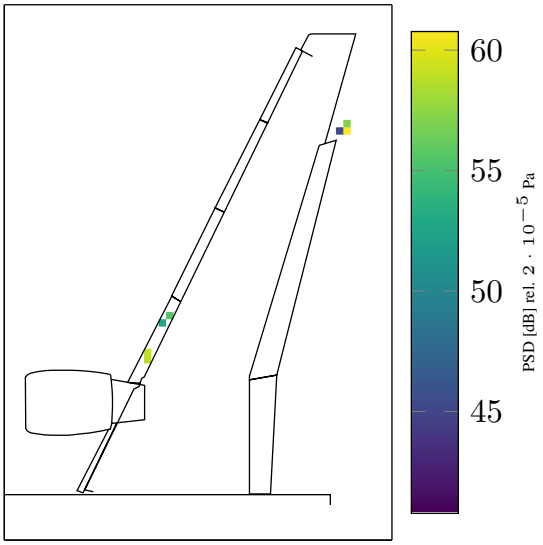

Figure 3: $\theta = 1,\ j^{\mathrm{Lep}} = 19, \alpha_{\mathrm{rel}} = 0.1$

Summary and conclusion

We considered the framework of generalized Tikhonov regularization for the inverse source problem in experimental aeroacoustics. Within this framework we presented a procedure how the Lepskiĭ principle can be used for the choice of the regularization parameter. As this principle allows for some abstract optimality results (cf. [4]) it is an interesting parameter choice rule from a mathematical point of view. This conference paper focused on the demonstration how to implement this principle for the aeroacoustic inverse problem, rather than evaluating the reconstruction results.

A more precise evaluation of the reconstruction results delivered by this principle certainly requires a comparison with other popular parameter choice strategies on a wider range of datasets. One challenge of such a comparison is that some strategies are bound to a specific penalty functional (i.e. purely L^2 regularization).

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