# Domain Decomposition with Fusion Frames

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## Abstract:
In computational acoustics the goal is to find a numerical solution of physical equations, like the Helmholtz equation. Traditionally, this is done by the boundary element or finite element method. To reduce the computational load the operator equation is split. The domain decomposition methods solves a boundary value problem by splitting it into smaller boundary problems on subdomains. It is vital to find the most applicable and tractable overlapping or non-overlapping domain decomposition for dividing a large problem to smaller one. One of the most important problems in this method is the splitting the given space and finding the global solution from the local ones, in particular in the overlapping domain case. In this paper, we approach this problem by using fusion frames, therefore allowing overlapping. At least in a mathematical setting fusion frames allows to find more numerically tractable approach on subdomains and combine the solution on the subspaces to get the global solution. We can apply this idea for operator equations and present several discretizations by fusion frames and their duals and analyze their numerical efficiency.

## Fusion frames
Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{w_i\}_{i\in I}$ be a family of weights, i.e. $w_i > 0, i \in I$. The sequence $W = (W_i, w_i)$ is called a fusion frame for $\mathcal{H}$ if there exist constants $0 < A_W \le B_W < \infty$ such that

$$A_W\|f\|^2 \le \sum_{i\in I} w_i^2\|\pi_{W_i}f\|^2 \le B_W\|f\|^2, \qquad (f \in \mathcal{H}).$$

The constants $A_W$ and $B_W$ are called *fusion frame bounds*. If the upper bound is fulfilled, we call $(W_i, w_i)$ a *Bessel fusion sequence*. A fusion frame $(W_i, w_i)$ is said to be a *fusion orthonormal basis* for $\mathcal{H}$ if $\mathcal{H}$ is the orthogonal sum of the subspaces and it is called a *Riesz decomposition* of $\mathcal{H}$ if for every $f \in \mathcal{H}$ there is a unique choice of $f_i \in W_i$ such that $f = \sum_{i\in I} f_i$. Moreover, we call a fusion frame *fusion Riesz basis* if it is Riesz decomposition.

The *synthesis operator* $T_W : \sum_{i\in I}\bigoplus W_i \to \mathcal{H}$ for a Bessel fusion sequence $(W_i, w_i)$ is defined by $T_W(\{f_i\}_{i\in I}) = \sum_{i\in I} w_i f_i$. The adjoint operator $T_W^* : \mathcal{H} \to \sum_{i\in I}\bigoplus W_i$ which is called the *analysis operator* is given by $T_W^* f = \{w_i\pi_{W_i}f\}_{i\in I}$.. For every fusion frame $W$, the *fusion frame operator* $S_W : \mathcal{H} \to \mathcal{H}$, which is defined by $S_W f = T_W T_W^* f = \sum_{i\in I} w_i^2\pi_{W_i}f$, is bounded (with bound $B_W$), invertible and positive [5, 8]. A Bessel fusion sequence $(V_i, v_i)$ is a dual of fusion frame $W = (W_i, w_i)$ if and only if $T_V\phi_{VW}T_W^* = \mathsf{id}_{\mathcal{H}}$, where the bounded operator $\phi_{VW} : (\sum_{i\in I}\bigoplus W_i)_{\ell^2} \to (\sum_{i\in I}\bigoplus V_i)_{\ell^2}$ is given by $\phi_{VW}(\{f_i\}_{i\in I}) = \{\pi_{V_i}S_W^{-1}f_i\}_{i\in I}$.

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{w_i\}_{i\in I}$ a family of weights. We say that $(W_i, w_i)$ is a *fusion Riesz basis* for $\mathcal{H}$ if $\overline{\text{span}}\{W_i\} = \mathcal{H}$ and there exist constants $0 < C \le D < \infty$ such that for each finite subset $J \subseteq I$

$$C\sum_{j\in J}\|f_j\|^2 \le \|\sum_{j\in J} w_j f_j\|^2 \le D\sum_{j\in J}\|f_j\|^2, \qquad (f_j \in W_j).$$

## Matrix Representation for Bessel Fusion Sequences
Assume that $W = (W_i, w_i)$ and $V = (V_i, v_i)$ are Bessel fusion sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Using the tensor product of operators, for any operator $O \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ we can define a matrix operator in $\mathcal{B}\left(\sum_{i\in I}\bigoplus V_i, \sum_{i\in I}\bigoplus W_i\right)$ by

$$\left(\mathcal{M}^{(W,V)}(O)\right)_{j,i} = w_j v_i \pi_{W_j} O \pi_{V_i}, \qquad (i, j \in I). \quad (1)$$

**Theorem 0.1** *Let $W = (W_i, w_i)$ and $V = (V_i, v_i)$ be Bessel fusion sequences in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with bounds $B_W$ and $B_V$, respectively.*

1. *Let $O : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Then the matrix of operators (1), denoted by $\mathcal{M}^{(W,V)}(O)$, defines a bounded operator from $\sum_{i\in I}\bigoplus V_i$ to $\sum_{i\in I}\bigoplus W_i$ with*

$$\left\|\mathcal{M}^{(W,V)}(O)\right\| \le \sqrt{B_W B_V}\,\|O\|.$$

*As an operator $\sum_{i\in I}\bigoplus V_i \to \sum_{i\in I}\bigoplus W_i$*

$$\mathcal{M}^{(W,V)}(O) = T_W^* O T_V = (T_W^* \otimes T_V^*)(O).$$

*So, $\mathcal{M}^{(W,V)} : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{B}\left(\sum_{i\in I}\bigoplus V_i, \sum_{i\in I}\bigoplus W_i\right)$ is a well-defined bounded operator.*

2. *On the other hand, let $M$ be an infinite matrix of operators defining a bounded operator from $\sum_{i\in I}\bigoplus V_i$ to $\sum_{i\in I}\bigoplus W_i$, $(Mf)_i = \sum_k M_{i,k}f_k$. Then the operator $\mathcal{O}^{(W,V)}$ defined by*

$$\mathcal{O}^{(W,V)}(M) := T_W M T_V^* = (T_W \otimes T_V)(M)$$

*is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ with*

$$\left\|\mathcal{O}^{(W,V)}(M)\right\| \leq \sqrt{B_W \cdot B_V}\,\|M\|\,.$$

*So, $\mathcal{O}^{(W,V)} : \mathcal{B}\left(\sum_{i\in I}\bigoplus V_i, \sum_{i\in I}\bigoplus W_i\right) \to \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ is a well-defined bounded operator.*

As an easy consequence of the properties of the tensor products of operators we get:

**Corollary 0.2** *Let $W = (W_i, w_i)$ and $V = (V_i, v_i)$ be Bessel fusion sequences. Then we have the following properties:*

1. *The operator $\mathcal{M}^{(W,V)} = T_W^* \otimes T_V^*$ is surjective if and only if $T_W^*$ and $T_V^*$ are surjective (i.e. $W$ and $V$ are Riesz sequences).*
2. *The operator $\mathcal{M}^{(W,V)} = T_W^* \otimes T_V^*$ is injective, if and only if $T_W^*$ and $T_V^*$ is injective (i.e. $W$ and $V$ are complete).*

Similar to the Hilbert frame case [3, 4] we can show formulas for the inverse of the matrix induced by an operator. They are not just a straightforward generalization, because of the interesting situation for fusion duals.

In the following, we compute the inverse of the matrix induced by the operator $O$ with respect to fusion Riesz bases $W$ and $V$. First we give the following proposition.

**Proposition 0.3** *Let $(W_i, w_i)$ be a fusion frame in $\mathcal{H}$. The following are equivalent:*

1. *$W$ is a fusion orthonormal basis.*
2. $\mathcal{M}^{(W^1,W^1)}(\mathsf{id}_{\mathcal{H}}) = \mathsf{id}_{\sum_{i\in I}\bigoplus W_i}$.
3. $\mathcal{M}^{(W,W)}(\mathsf{id}_{\mathcal{H}}) = \phi_{WW}^{-1}$.
4. *For all invertible operators $O \in \mathcal{B}(\mathcal{H})$, $\mathcal{M}^{(W,W)}(O)$ is invertible and $\left(\mathcal{M}^{(W,W)}(O)\right)^{-1} = \mathcal{M}^{(W,W)}(O^{-1})$.*
5. *There is an operator $O \in \mathcal{B}(\mathcal{H})$ commuting with $S_W$, such that $\left[\mathcal{M}^{(W,W)}(O)\right]^{-1} = \mathcal{M}^{(W,W)}(O^{-1})$.*

Finally see that all results stay true for the lesser conditions of assuming injective (one-to-one) respectively surjective (onto).

**Theorem 0.4** *Let $W = (W_i, w_i)$, $V = (V_i, v_i)$ be fusion frames in $\mathcal{H}$. Then the following are equivalent:*

1. *$W$ and $V$ are fusion Riesz bases.*
2. *$\mathcal{M}^{(W,V)}$ is onto.*
3. *$\mathcal{M}^{(W,V)}$ is invertible and $\left(\mathcal{M}^{(W,V)}\right)^{-1} = \left(S_W^{-1}\otimes S_V^{-1}\right)\mathcal{O}^{(W,V)}$.*
4. *There exists an invertible operator $O \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{M}^{(W,V)}(O)$ is invertible and*

$$\left(\mathcal{M}^{(W,V)}(O)\right)^{-1} = \mathcal{M}^{(V,W)}(S_V^{-1}\otimes S_W^{-1})(O^{-1}).$$

## Solving Operator Equations

Consider the operator equation

$$Of \;=\; g, \tag{2}$$

where $O \in \mathcal{B}(\mathcal{H})$. By the discussion above, for the operator equation (2) we can introduce the following systems of linear equations where $V$ is a dual fusion frame of $W$.

$$\mathcal{M}^{(W,V)}(O)\phi_{VW}T_W^* f = T_W^* g \tag{3}$$
$$\mathcal{M}^{(W,V)}(OS_V^{-1})T_V^* f = T_W^* g \tag{4}$$
$$\mathcal{M}^{(V,W)}(OS_W^{-1})T_W^* f = T_V^* g \tag{5}$$
$$\mathcal{M}_{\otimes}^{(W,V)}(S_W^{1/2}OS_V^{-1/2})T_V^* f = T_W^* g \tag{6}$$
$$\mathcal{M}_{\otimes}^{(W,V)}(S_W^{1/2}OS_V^{1/2})\phi_{VW}T_W^* f = T_W^* g \tag{7}$$
$$\mathcal{M}_{\otimes}^{(V,W)}(S_V^{1/2}OS_W^{-1/2})T_W^* f = T_V^* g. \tag{8}$$

Note that, for numerical computations, the above linear systems can be solved. It is natural to ask which of the above representations is numerically more efficient. Between (3), (4) and (5), the first equation is much more favorable. Indeed, the inverse of a frame operator appears in all of them, if only implicitly (in (3)).

We propose to solve the following system instead

$$\mathbf{A}x = b \text{ for } \mathbf{A} = \left[\mathcal{M}^{(W,W)}(O)\right], x = \left[\phi_{VW}^* T_V^* f\right] \text{ and } b = T_W^* g, \tag{9}$$

where $V$ is any dual of $W$. This is, of course, not unique. Solving this system results in determining $x$. We get a solution of (2), i.e. $Of = g$, in the original space by $f = T_W x$. It is worthwhile to mention that, for every dual of $W$, we can get a solution of (9), but it is only involved implicitly. In particular, no inversion of a frame operator is necessary here (compare to the Hilbert frame setting in [2]). This is a reason why we have not used $\mathcal{M}_{\otimes}^{(W,W)}$ (which is for example more straight-forward for the reconstruction of an operator, or the computation of the matrix representation of a product of operators, as it involves the inverse of the frame operator. This makes perfect sense for fixed subspaces, as the inverse can then be calculated a-priori offline. It is certainly not effective for an adaptive approach, see e.g. [6].

So, any solution of (9) is equivalent to the solution of the original problem (2). The solution on the subspaces can be found by a matrix representation using the local frames in the sense of [1]. So in total this gives us a scheme to solve the operator equation on the sub-spaces and then combine all the local solutions to get a global solution. But note that here any of the involved operator matrix might not be well conditioned and is, in general, full. In particular, that means that all combinations of subspaces have to be included.

The next step would be to find a more numerically tractable approach for example a Richardson iteration [12] using equation (9) to solve (2). We can formulate it as

$$x^{n+1} = x^n - \lambda\left(\mathbf{A}x^n - b\right). \tag{10}$$

The other formulations (3) to (8) would correspond to a preconditioning. One could use other preconditioning approaches like the more traditional ones, now applied to the operator matrix setting. Let us suppose that the operator $O$ is coercive (a common assumption e.g. in the analysis of elliptic PDEs), so that

$$\mathbf{A}_{i,i} = \left[\mathcal{M}^{(W,W)}(O)\right]_{i,i}$$

has a bounded inverse. Then we can use the Jacobi preconditioning approach transformed to the block matrix setting. Let the operator $\mathbf{P} : \sum_{i\in I} \bigoplus W_i \to \sum_{i\in I} \bigoplus W_i$ be given by $\mathbf{P} = \oplus \left(\mathbf{A}_{i,i}\right)^{-1}$, and the solve the equation

$$\mathbf{PA}x = \mathbf{P}b, \tag{11}$$

again by the iteration above. As we cannot guarantee that $x^n$ is in the range of $\phi^*_{VW}T^*_V$, this iteration will probably sum up error terms in the complement of this range, which would clutter up the algorithm. As in [11] one could attack this problem also in the subspace setting by a (not necessarily orthogonal) projection, so we suggest to do the following iteration:

$$x^{n+1} = \phi^*_{VW}T^*_V T_W \ \left[x^n - \lambda\left(\mathbf{PA}x^n - \mathbf{P}b\right)\right]. \tag{12}$$

It is worthwhile to mention that if $x$ is a solution of (11), then $\dot{x} := \phi^*_{VW}T^*_V T_W x$ is also. Indeed,

$$\begin{aligned}\mathbf{PA}(\phi^*_{VW}T^*_V T_W x) &= \mathbf{P}T^*_W O T_W \phi^*_{VW}T^*_V T_W x \\ &= \mathbf{P}T^*_W O T_W x \\ &= \mathbf{PA}x = \mathbf{P}b.\end{aligned}$$

Moreover, $f = T_W x = T_W(\phi^*_{VW}T^*_V T_W x)$ and by above is a solution of the original equation (2). To speed up the iteration one could do the second step of "projection" not at every step, but only every $N$-step (an idea used for example in [10]), as $x$ and $\dot{x}$ are both solutions.

Alternatively, to avoid that problem, one could do a Richardson iteration on the Hilbert space side. Still we can use the approach above, to find a good preconditioning. Set

$$\mathcal{P} = \mathcal{O}^{(W,W)}\left(\mathbf{P}\right) = \mathcal{O}^{(W,W)}\left(\oplus\left(\mathbf{A}_{i,i}\right)^{-1}\right).$$

(We could, of course, also use other combinations of $W$,$V$ and $\phi_{W,V}$.) In this case, we use the iteration

$$f^{(n+1)} = f^{(n)} - \lambda\left(\mathcal{P}O f^{(n)} - \mathcal{P}g\right).$$

This corresponds to the *additive Schwarz method*, which is a block Jacobi approach [7]. Theorem 2 in [9] is then just a consequence of the comments given above.

A more detailed treatment of all these question will be the topic of future research.

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